Evaluation of velocity measurements with a video camera

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Contents

1 Introduction

Gravity currents with balls of uniform sizes are carried out in chutes and open, unchanneled slopes. As balls, very light pingpongballs as well as heavy golfballs are used. This text deals with the measurement of the vertical velocity ditribution with a video camera, which is positioned perpendicular above the flow. The theoretical background of such flows will not be discussed here.

1.1 Measurement setup

The video camera is set perpendicular to the ground. The measurement principle is as follows: The balls close to the lens of the camera appear bigger than those which are further away. Because all balls in one flow have an uniform diameter, the distance from the camera to a ball can be calculated by measuring the size of a ball in the video picture. In the same way, by measuring the distance the ball moves within a certain time, the velocity of the ball can be calculated. Not only the two dimensional velocity vector, parallel to the ground, can be calculated, but also the component in the z-direction, by taking into account the different ball heights before and after this time interval. This time can either be the time interval of two successive frames, t_i ("two picture measurement"), or the time of the shutter speed, t_s ("one picture measurement"). In the first case, the ball must be seen in both frames, in the second one, the entire path of the ball, looking as a long, rather faint line of the width of one ball diameter must be recognized and measured.

The height h_{cam} of the camera (resp. lens) over the ground is dependent on the opening angle of the camera lens, on the maximum ball velocity v_{max}, the maximum flow height h_{max} and the principle of measurement (one or two picture measurement). In order to get a reasonable resolution of the ball height, the camera should be very close to the flow (see figure 1).

1.2 Notation

A drawing of the setup with some basic notations can be seen in figure 2.

d_b	diameter of ball,
z_b	height over ground of ball,
$\vec{v}(z)$	velocity of ball at height z over ground,
u	velocity component in flow direction (x),
v	velocity component in y - direction,
w	vertical velocity component (z - direction),
u_{max}	maximum velocity of ball, in flow direction,
h_{max}	maximum flow height,
h_{len}	distance ground - lens,
d_{lb}	$= h_{len} - h_b$, distance lens - ball,
d_{min}	$= h_{len} - h_{max}$, minimum distance lens - maximum flow height,
m, b	geometrical characteristics of lenses,
$l(d_{lb})$	$= p_{max} \cdot m(d_{lb} + b)$, length of representation at distance d_{lb},
l_{min}	minimum required length of representation at maximum flow height h_{max} (dependent of v_{max} and principle of measurement),

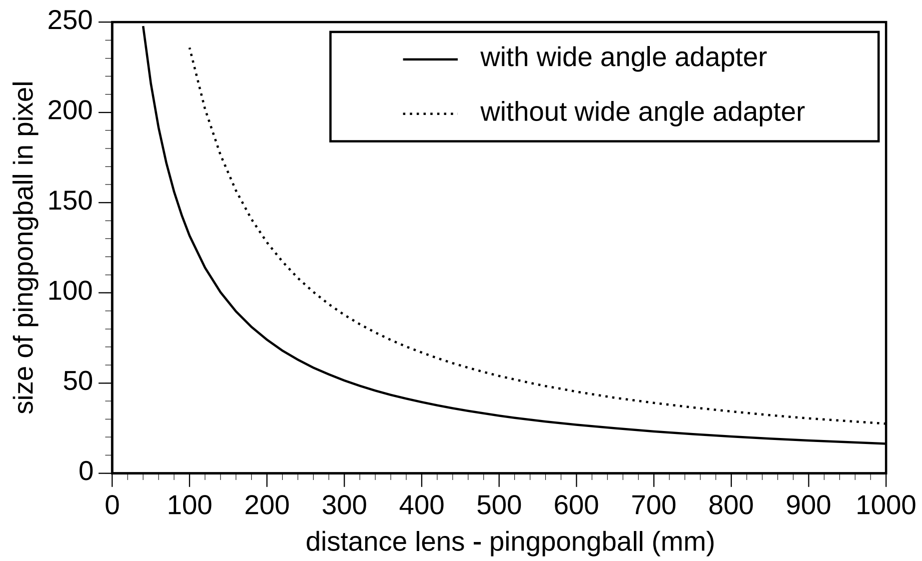

Figure 1: Size of a pingpongball in pixels in relation to the distance to the lens. The maximum wide angle position of the lens, without and with additional wide angle adapter is presented. The closer the ball to the lens, the steeper the curve and the better is the height resolution. The size in pixel is referred to a picture size of 512 pixel (long side).

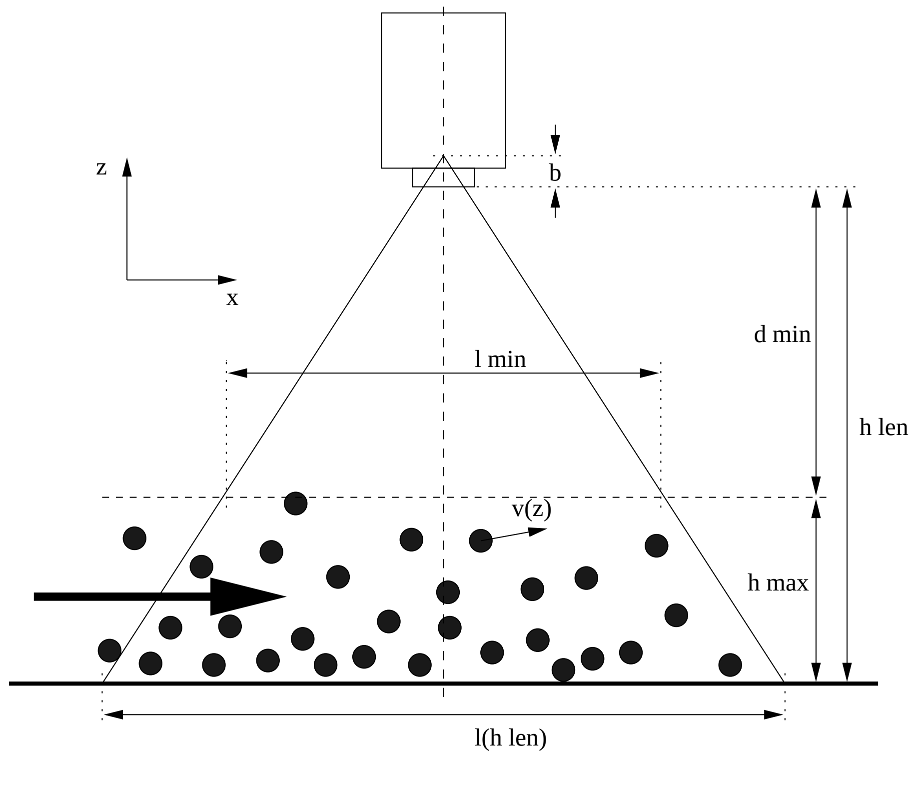

Figure 2: Schematic view of the measurement setup. The notations are explained in the text.

t_i	frame interval of camera,
t_s	shutter speed of camera,
p_{max}	number of pixels in the length side of the video frame,
$p(d_{lb})$	frame fraction (from length side) of ball at distance d_{lb} from lens.

These notations are related to a camera, which is fixed perpendicular to the ground, with the long frame side in the flow direction. However, there might be the situation, that it is not possible to fix it in that way, either that the camera axis is not vertical to the ground or the orientation of the camera frame is not parallel to the flow direction. This has to be taken into account by the evaluation.

1.3 Coordinate systems

The coordinate system of the camera frame in pixels (let's say $p_x - p_y$ - plane or p-plane with $p(p_x, p_y)$ as a point on it) shall be defined with zero in the centre of the frame with the two axis p_x and p_y parallel to the edges, with p_x somewhat in the in main flow direction, i.e., along the long side of the frame (figure 3).

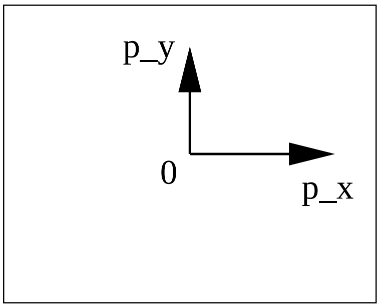

Figure 3: Coordinate system of the video frame, defined as the $p_x - p_y$ - plane (p-plane). It can be regarded as the two-dimensional pixel frame. The origin, 0, is situated in the centre of the frame. The t-plane, where the correction of the lens-distortion is taken into account, is very similar.

Basically equivalent to the p-plane is the system with the correction of the distortion, the $t_x - t_y$ - plane or, shorter, the t-plane, with $t(t_x, t_y)$ as a point on it. It has the same origin and orientation, but the axes have other scalings. The unit shall be called *newpixel*. A second coordinate system of the camera can be defined as a three-dimensional space (say $x_c - y_c - z_c$ - space or c-space, with $c(x_c, y_c, z_c)$ as a point in the space) with the same orientation as the pixel-coordinates. The units of this system shall be the usual length units, as at the ground coordinate system. The origin 0 shall be in the centre of the video frame, i.e., in the focus of the lens system. In spite of the same orientation as the $p_x - p_y$ - plane, the z_c - direction to the ground is *negatif* (figure 4). This is in order to keep the same main orientation as the coordinate system of the measurement setup, with usual units and $x - y - z$ - notation, with the x - axis in the main flow direction and w upwards. The origin of the $x - y - z$ - space (x-space, with $g(x, y, z)$ as a point in it) shall be at the base of the vertical line from the ground plane ($x - y$) to the focus of the video camera.

2 Calibration of video geometry

The calibration includes two parts: The properties of the lens system at the used focal distance with the distortion towards the corners of the frame, and secondly

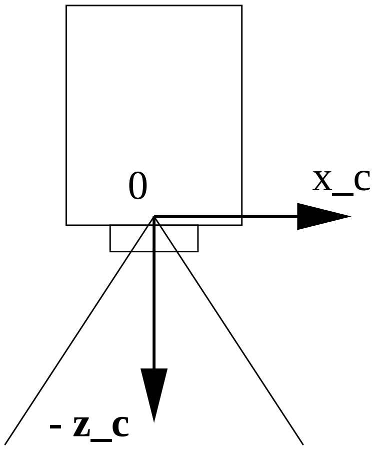

Figure 4: $x_c - y_c - z_c$ - coordinate system (c-space) of the video camera. The origin is defined in the focus of the lens. The orientation is the same is in the pixelframe (figure 3) and in the ground $x - y - z$ - coordinate system, therefor the z_c - direction towards the ground is negatif. The units are the common length units.

the setup of the camera within the system of the experiment. The first part can be made in the laboratory, and is valid as long as the lens system is unchanged and a constant focal distance is used. The second calibration has to be made at each new experimentel setup, in order to get the orientation of the camera.

2.1 Properties of the lens system

For the calibration of the lens system (at a fixed setting) two mappings are necessary:

1. Distortion f_{dt}: $p \longmapsto t$, $f_{dt}(p(p_x, p_y)) = t(t_x, t_y)$.
2. Focus f_{foc}: $t \longmapsto c$, $f_{foc}(t(t_x, t_y)) = c(x_c, y_c, z_c)$.

2.1.1 Distortion

The calibration of the distortion, f_{dt}, can be made with a grid, which is fixed some distance away from the lens. This grid must be positioned vertical to the axis of the lens. This can be easily made by using a mirror, parallel on the plate with the grid. The camera is now fixed in a position, where the mirror image of the lens is exact in the middle of the whole frame. The orientation and the origin of the grid coordinates must be the same as on the p- and t-plane The pixelcoordinates of different points of the grid can be determined (p-plane, figure 5). Either one can find a closed solution for the mapping f_{dt}, or a grid mapping and interpolation can be used.

The latter may look like the follows: In the p-plane, the coordinates of the points of intersections of the grid lines are measured (such as point p in figure 5). These points can be easily mapped to the t-plane by making the lines parallel and vertical (in the figure, point p is mapped to t). To get the mapping $f_{dt}(m(p)) = m(t)$ of any measurement point $m(p)$ in the p-plane, the grid field, where $m(p)$ is located is searched. The coordinates of the corners of the field are known in the t-plane, therefore, $m(t)$ can be determined by keeping the same location within the the grid field, now in the t-plane.

The location within the grid can be calculated like the following, assuming straight grid lines (see figure 6; actually, they are bent): The line $\overline{Ma\,Mc}$

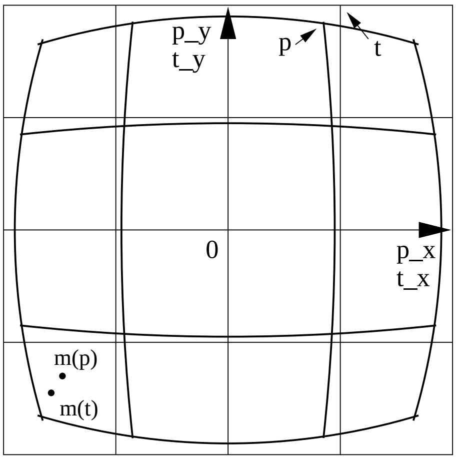

Figure 5: Schematical presentation of the distortion. The grid with the parallel and vertical lines is seen on the p-plane as a distorted grid (thick lines). The mapping f_{dt} brings the point $p(p_x, p_y)$ in video frame (in pixel-units) to $t(t_x, t_y)$ (in newpixel-units). In the t-plane (thin lines), all in the original parallel lines must be parallel again. The p- and the t-planes have the same origin and the same orientations.

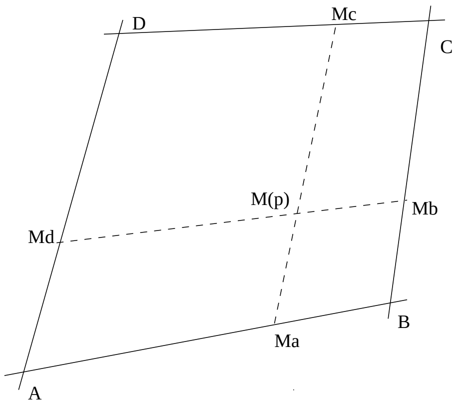

Figure 6: Interpolation from the p-frame to the t-frame. Here, in the p-frame, straight grid lines are assumed. Then, the location of $M(p)$ within the grid is given with certain ratios, which can be used for the calculation of M in the t-frame.

goes through $M(p)$ in a way, that $\overline{A\,Ma}/\overline{A\,B} = \overline{D\,Mc}/\overline{D\,C}$. For the ratio $\overline{Md\,M}/\overline{Md\,Mb}$ on the analogous line between $\overline{A\,B}$ and $\overline{D\,C}$, the same ratio is valid: $\overline{Md\,M}/\overline{Md\,Mb} = \overline{A\,Ma}/\overline{A\,B}$. Thus, calculating this ratio, it can be used in the t-frame to calculate $M(t)$.

A more accurate method uses a two-dimensional spline interpolation, which is carried out from the rectangular t-frame to the distorted p-frame. With an iteration, $f_{dt}^{-1}(m(t))$ comes as close to $m(p)$, the starting value, as wished.

2.1.2 Focal length

The calibration of the focus distance, i.e., of the mapping f_{foc}: $t \longmapsto c$, can be made with a length scale, fixed at different distances from the lens, vertical to the lens axis in the middle of the frame, in the direction of the t_x-axis. Here, the same grid as above can be used, but only the scale in one direction is of importance. At a distance D from the lens, the newpixels T of the length L on the scale are measured. At another distance D_2, the newpixels T_m of the length L_m are measured (see figure 7). To get same value T in newpixels at both distances D and D_2, it is

$$L_2 = L_m \frac{T}{T_m}\,. \tag{1}$$

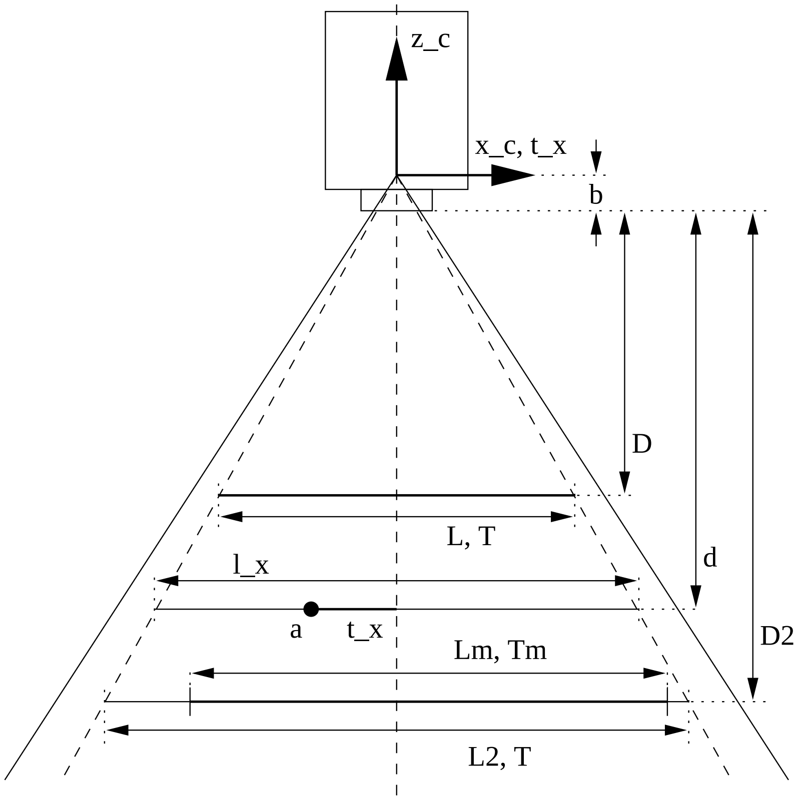

Figure 7: Sketch of the calibration of the coordinate system of the camera (c-space). Big letters refer to calibration values, b denotes the distance from the edge of the lens to the focus. The aim of the calibration is the mapping of a point (such as a) in the t-plane with a *given* distance d from the lens to a point in the c-space.

From the proportionality follows

$$\frac{L}{D+b} = \frac{L_2}{D_2+b} \tag{2}$$

and for b

$$b = \frac{L \cdot D_2 - L_2 \cdot D}{L_2 - L} \,. \tag{3}$$

For a point a with a coordinate t_x in the t-plane and a *given* distance d from the lens (i.e., d is the distance to the plane, where the point a is located on, and which is vertical to the axis of the lens), the new x-coordinate in the c-space, x_c, is

$$x_c = \frac{t_x}{T}\, l_x \,, \tag{4}$$

with l_x the equivalent to L and L_2, the length proportional to T newpixels. For l_x it is, as above,

$$l_x = \frac{L}{D+b} \cdot (d+b) \tag{5}$$

and therefore

$$x_c = \frac{t_x}{T}\, \frac{L}{D+b} \cdot (d+b) \,. \tag{6}$$

All in all, with

$$m = \frac{L}{T\,(D+b)} \tag{7}$$

and b from equation 3, the lens system with a fixed focus is defined. Because of the radial symmetrie of the lens, m, b and the above calculation for the x-coordinate is valid for the y-coordinate, too. The direct dependances of m and b on the calibration values L, D and T and L_m, D_m and T_m look like the following:

$$b = \frac{L\, D_m\, T_m - L_m\, D\, T}{L_m\, T - L\, T_m}, \tag{8}$$

$$m = \frac{L_m\, T - L\, T_m}{T\, T_m\, (D_m - D)}. \tag{9}$$

Therefore, the mapping f_{foc}: $t \longmapsto c$ of a point $a(t_x, t_y)$ in the t-plane with a distance d from the lens is given with

$$x_c = t_x \cdot m\,(d + b)\,, \tag{10}$$

$$y_c = t_y \cdot m\,(d + b)\,, \tag{11}$$

$$z_c = -(d + b)\,, \tag{12}$$

with b and m from equations 9 and 9, respectively.

Two remarks can be added:

1. The constant m includes both, the lens characterization *and* the maximum number of pixel p_{max} in the frame.

2. With the maximum number of pixels p_{max} in the length of the frame, the "view"-length $l(d)$ at the distance d from the lens is given: $l(d) = p_{max} \cdot m(d+b)$.

2.2 Setup of the camera in the system of the experiment

This calibration has to be made at each experiment, where the camera is new installed or their position is changed. The procedure is as follows: A scale is fixed at the ground, the direction is the main flow direction and defines the x-axis of the x-space. The scale must go through the point, where the vertical line to the focus of the camera meets the ground plane (V). Again, this can be easily fixed with a mirror, as stated above. Beside of the point V (see figures 8 and 9), two other points on the scale, towards both sides of the frame, shall be fixed while looking through the camera. Their names shall be A and B and the measured distance $\overline{AB}$ is ΔD (ΔD remains constant in the x- and c-space). In the pixelframe, the p-plane, the corresponding coordinates p_V, p_A and p_B can be determined.

The aim of this calibration is now to define the mapping from the c-space to the x-space. This includes the determination of the vertical distance h_f from the focus to the ground plane and the different rotation angles. In the case that the axis of the lens of the camera is fixed perpendicular to the ground and the x-axis of the p-, resp. t-plane and the c-space is in the direction of the main flow, no rotation to the x-space occurs and only the vertical distance h_f has to be determined.

The procedure looks like the following. The points p_V, p_A and p_B are measured in the p-plane. With the mapping f_{dt}: $p \longmapsto t$ we get t_V, t_A and t_B, all points should be on one line yet. The following mapping f_{foc}: $t \longmapsto c$ for these points can not be calculated, as long as the distances h_V, h_A and h_B are not known (see figure 9). Let's imagine a rotation of the x-space around the y-axis of the c-space, the rotation

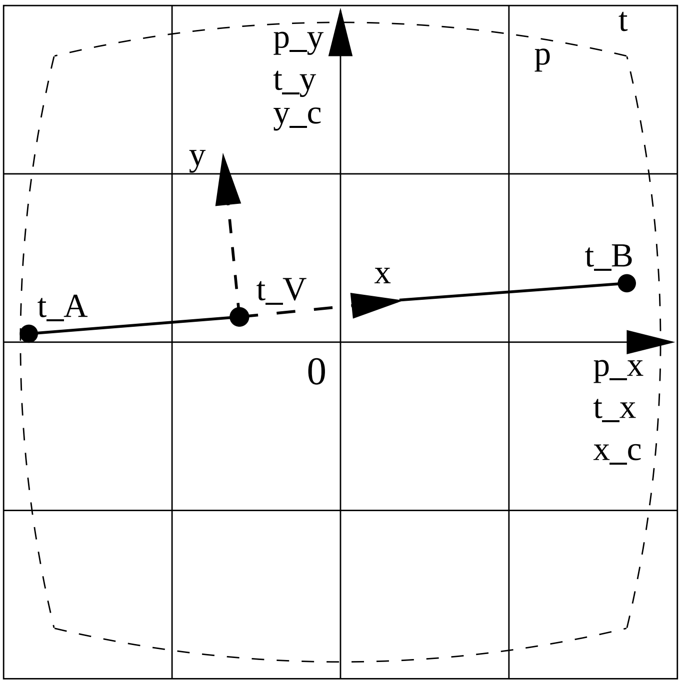

Figure 8: Ground sketch of the calibration of the camera in the system of the experiment. In the p-plane the coordinates of V, A and B, are measured, all of them points in the ground plane, whereas V is the point of intersection of the straight line, which is perpendicular to the ground plane and which goes to the focus of the lens. The direction from A to B determines the main flow direction and the x-axis of the x-space. Since p_V is not in the origin of the p-plane, the $x-y$-plane in the x-space is not parallel to the $x_c - y_c$-plane of the c-space. As a consequence, different rotations are needed.

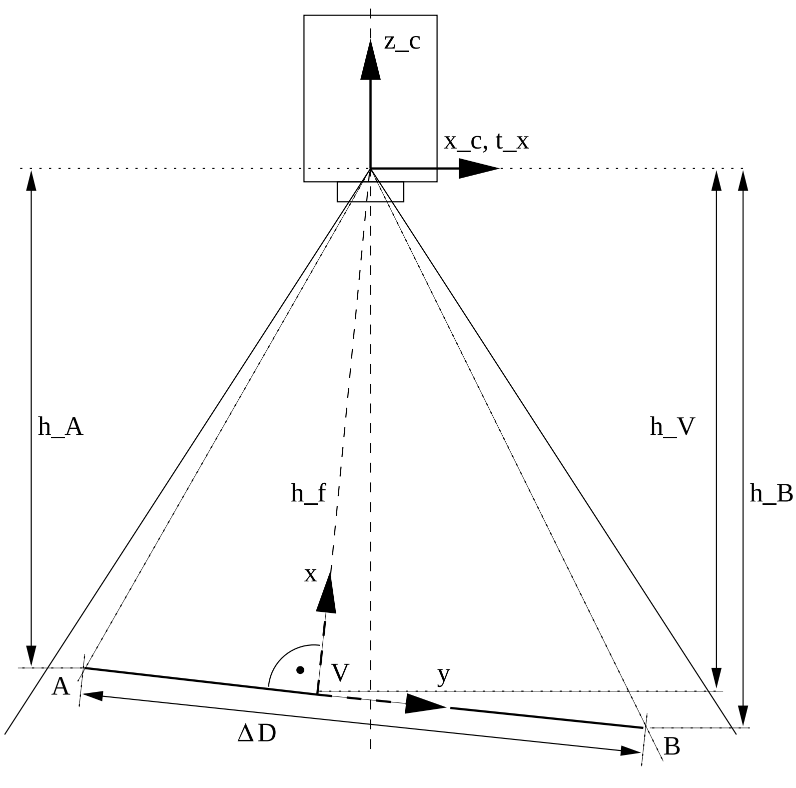

Figure 9: Side view of the calibration of the camera in the system of the experiment. Beside of the different rotation angles, the distance h_f from V to the focus of the lens must be determined.

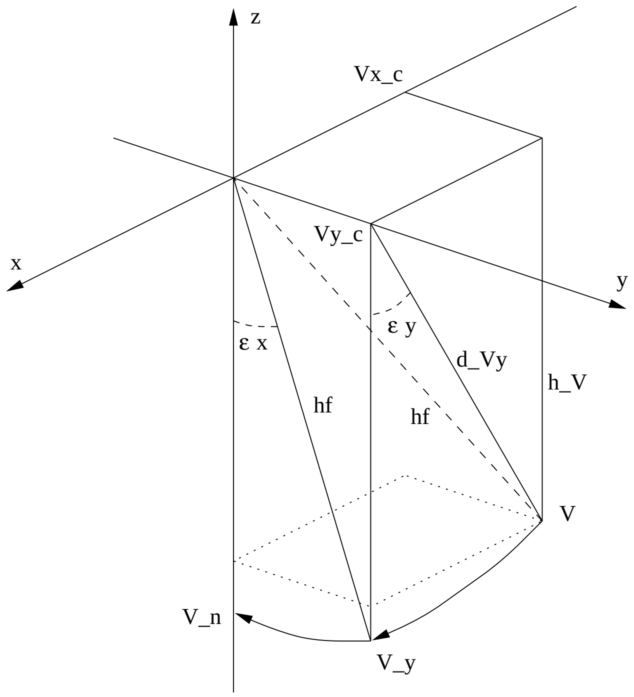

Figure 10: Sketch of the rotations of V to V^y and to V^n. V is the point of intersection between the ground plane and the line from the focus perpendicular to the ground. After the rotations, V^n is situated on the z-axis of the c-space.

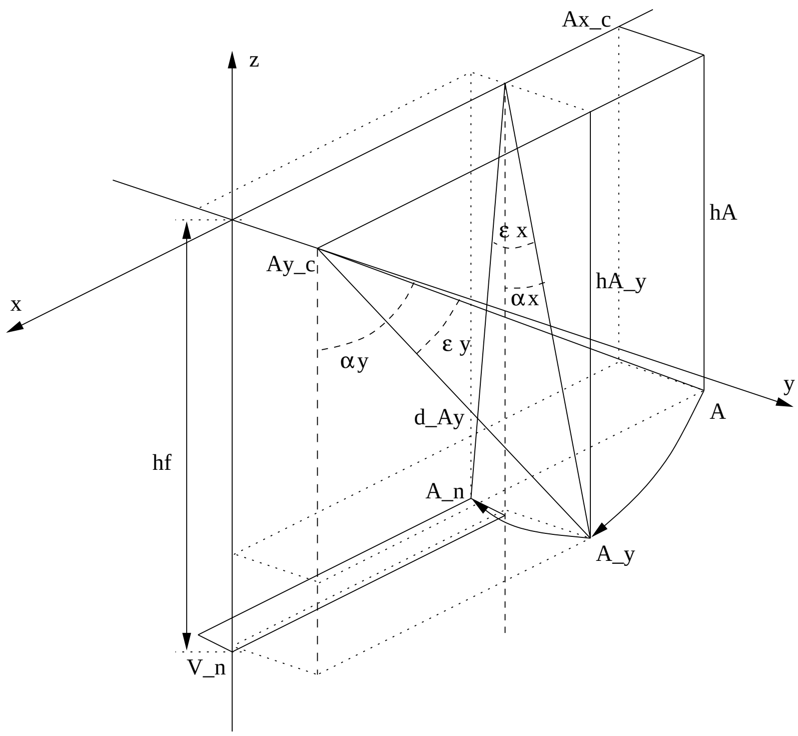

Figure 11: Sketch of the rotations of A to A^y and to A^n. A lies on the ground plane and has after the rotations the same z-coordinate as V^n, h_f.

angle ϵ_y is given in way, that the new x-coordinate of the point V is 0. This rotation is only related the the x-coordinates and leaves the y-values unchanged.

For the point V, following the mapping from t to c (equation 10), it is

$$Vx_c = t_{Vx} \cdot m \cdot h_V \,, \tag{13}$$

h_V, unknown yet, is the distance from V to the plane, which goes through the focus of the camera and which is perpendicular to the axis of the lens (z-axis of the c-space). The rotation angle ϵ_y is given with (see figure 10)

$$\tan \epsilon_y = \frac{Vx_c}{h_V} = t_{Vx} \cdot m \,. \tag{14}$$

After the rotation the x-coordinate of V is zero. For any point $A \neq V$ in the ground plane (see figure 11) it is

$$Ax_c = t_{Ax} \cdot m \cdot h_A \,. \tag{15}$$

The angle α_y is given as above with

$$\tan \alpha_y = \frac{Ax_c}{h_A} = t_{Ax} \cdot m \,. \tag{16}$$

After the rotation with the angle ϵ_y it is

$$\tan(\alpha_y - \epsilon_y) = \frac{Ax_c^y}{h_A^y} \,, \tag{17}$$

h_A^y is the new distance to the plane, which goes through the focus and which is perpendicular to the axis of the lens. If d_{Ay} is the distance from A and A_y to the y-axis, it is

$${d_{Ay}}^2 = {h_A}^2 + (Ax_c)^2 = (h_A^y)^2 + (Ax_c^y)^2 \,. \tag{18}$$

The new x-coordinates in the c-space and the t-plane, respectively, are given with

$$Ax_c^y = \tan(\alpha_y - \epsilon_y) \cdot h_A^y \tag{19}$$

and

$$t_{Ax}^y = \frac{Ax_c^y}{m \cdot h_p^y} = \frac{\tan(\alpha_y - \epsilon_y)}{m} \,. \tag{20}$$

Since ϵ_y is given with equation 14, t_{Ax}^y is determined, but not Ax_c^y because of the unknown h_A^y (equation 19 is not determined).

The next step is the analogous rotation around the x-axis of the c-space. This rotation angle ϵ_x is given with the condition, that the new y-coordinate of V is 0. This rotation affects only y-values. Again it is

$$Vy_c = t_{Vy} \cdot m \cdot h_V \,, \tag{21}$$

the rotation is not related anymore to the distance h_V but

$$\tan \epsilon_x = \frac{Vy_c}{d_{Vy}} = \frac{t_{Vy}\, m\, h_V}{d_{Vy}} \,, \tag{22}$$

d_{Vy} is the distance from V and V^y to the y-axis (analogous to d_{Ay}), and it is

$$ {d_{Vy}}^2 = {h_V}^2 + {Vx_c}^2 \,. \tag{23} $$

The relation between h_f, the wanted distance from V to the focus, and h_V is given with

$$ {h_f}^2 = {h_V}^2 + {Vx_c}^2 + {Vy_c}^2 \,. \tag{24} $$

Together with the equations 13 and 14, it follows for h_V and d_{Vy}

$$ {h_V}^2 = {h_f}^2 \frac{1}{1 + {t_{Vx}}^2 m^2 + {t_{Vy}}^2 m^2} \,, \tag{25} $$

$$ {d_{Vy}}^2 = {h_f}^2 \frac{1 + {t_{Vx}}^2 m^2}{1 + {t_{Vx}}^2 m^2 + {t_{Vy}}^2 m^2} \,. \tag{26} $$

For the rotation angle ϵ_x (equation 22) we get therefore

$$ \tan \epsilon_x = \frac{t_{Vy} \cdot m}{(1 + {t_{Vx}}^2 m^2)^{\frac{1}{2}}} \,, \tag{27} $$

i.e., the rotation angle ϵ_x is independent on the distance h_f. For the point A it is

$$ Ay_c = t_{Ay} \cdot m \cdot h_A \,, \tag{28} $$

the angle between $A^y = (Ax_c^y, Ay_c)$, the x-axis and the $x - z$-plane is

$$ \tan \alpha_x = \frac{Ay_c}{h_A^y} = t_{Ay} \cdot m \frac{h_A}{h_A^y} \,. \tag{29} $$

For h_a and h_A^y it is

$$ \frac{h_A}{d_{Ay}} = \sin(\frac{\pi}{2} - \alpha_y) = \cos \alpha_y \,, \tag{30} $$

$$ \frac{h_{Ay}}{d_{Ay}} = \sin(\frac{\pi}{2} - (\alpha_y - \epsilon_y) = \cos(\alpha_y - \epsilon_y) \,. \tag{31} $$

Therefore, the angle α_x is given with

$$ \tan \alpha_x = t_{Ay} \cdot m \, \frac{\cos \alpha_y}{\cos(\alpha_y - \epsilon_y)} \,. \tag{32} $$

After the rotation it is

$$ \tan(\alpha_x - \epsilon_x) = \frac{Ay_c^n}{h_f} \,, \tag{33} $$

$$ Ay_c^n = \tan(\alpha_x - \epsilon_x) \cdot h_f \,, \tag{34} $$

$$ t_{Ay}^n = \frac{Ay_c^n}{m \cdot h_f} = \frac{\tan(\alpha_x - \epsilon_x)}{m} \,. \tag{35} $$

Here, we have h_f, since A^n lies on the ground plane and this is, after these two rotations, parallel to the $x-y$-plane and goes through V^n, which lies on the z-axis. The relation between h_f and h_A^y, which determines Ax_c^y in equation19 is given with the distance $d_{A^y x}$ from A^y to the x-axis:

$$\frac{h_f}{\cos(\alpha_x - \epsilon_x)} = d_{A^y x} = \frac{h_A^y}{\cos \alpha_x} . \tag{36}$$

With these two rotations around the y- and the x-axis of the c-space and with the angles ϵ_y and ϵ_x, given in equations 14 and 27, the ground plane with V is now parallel to the $x-y$-plane of the c-space. But the distance h_f from the ground plane to the focus is not known yet.

h_f is related to the measured length $\Delta D = \overline{AB}$. In newpixel units, the length is $\overline{t_A^n t_B^n}$, with equations 10 and 11 it is with $t_{Ax}^n = t_{Ax}^y$ and $t_{Bx}^n = t_{Bx}^y$:

$$\Delta D = \overline{t_A^n t_B^n} \cdot m \cdot h_f , \tag{37}$$

and with equation 35

$$\begin{aligned} \overline{t_A^n t_B^n}^2 &= (t_{Bx}^n - t_{Ax}^n)^2 + (t_{By}^n - t_{Ay}^n)^2 , \\ &= \left(\frac{\tan(\beta_y - \epsilon_y) - \tan(\alpha_y - \epsilon_y)}{m} \right)^2 \\ &\quad + \left(\frac{\tan(\beta_x - \epsilon_x) - \tan(\alpha_x - \epsilon_x)}{m} \right)^2 . \end{aligned} \tag{38}$$

Here, all angles are defined by the newpixel units of V, A and B, and, therefore, the whole system is determined.

The final step is a rotation around the z-axis of the c-space in order to get the angle ϵ_z of the flow direction (given with $\overrightarrow{AB}$) to the x-axis of the c-space. Because $Vx_c = Vy_c = 0$, ϵ_z is only dependend on A (or B):

$$\tan \epsilon_z = \frac{Ay_c^n}{Ax_c^n} = \frac{\tan(\alpha_x - \epsilon_x) \cdot h_f}{\tan(\alpha_y - \epsilon_y) \cdot h_A^y} . \tag{39}$$

The new coordinates of A are dependend on h_A^y and h_f, with the equation 36 it is finally

$$\tan \epsilon_z = \frac{\sin(\alpha_x - \epsilon_x)}{\tan(\alpha_y - \epsilon_y) \cos \alpha_x} . \tag{40}$$

Therefore, the last rotation around the z-axis is independent on h_f, too.

The same calculation can be made for the situation, where V is not situated on the straight line AB. This is even more conveniant for the procedure of the calibration.

2.2.1 Summary of the main equations

Rotation around the y-axis

$$\tan \epsilon_y = t_{Vx} \cdot m .$$

Rotation around the x-axis

$$\tan \epsilon_x = \frac{t_{Vy} \cdot m}{(1 + {t_{Vx}}^2 m^2)^{\frac{1}{2}}} .$$

Relevant angles for a point A on the ground plane

$$\tan \alpha_y = t_{Ax} \cdot m ,$$

$$\tan \alpha_x = t_{Ay} \cdot m \, \frac{\cos \alpha_y}{\cos(\alpha_y - \epsilon_y)} .$$

Rotation around the z-axis

$$\tan \epsilon_z = \frac{\sin(\alpha_x - \epsilon_x)}{\tan(\alpha_y - \epsilon_y) \cos \alpha_x} .$$

Distance h_f from V to the focus

$$h_f = \frac{\Delta D}{\overline{t_A^n t_B^n}^*} ,$$

$$(\overline{t_A^n t_B^n}^*)^2 = \Big(\tan(\beta_y - \epsilon_y) - \tan(\alpha_y - \epsilon_y)\Big)^2 + \Big(\tan(\beta_x - \epsilon_x) - \tan(\alpha_x - \epsilon_x)\Big)^2 .$$

2.2.2 Mapping from the system of the camera to the system of the experiment

This mapping consists of the three rotations around the axes of the c-space, the system of the camera, and a shifting along the x-axis of the x-space. Special attention has to be made at the definitions of the angles and the directions of the rotations. It follows, that the necessary rotations around the x-axis and the z-axis are opposite to the definitions of ϵ_x and ϵ_z. In the following calculation, the correct directions of this angles are already assumed, i.e., the rotations are always in the positive direction.

The point in the c-space shall be $\vec{c} = (x_c, y_c, z_c)$. The first step is a rotation around the y-axis, with the angle ϵ_y, the value of y_c remains untouched. The rotation is given with

$$\vec{c}^{\,y} = \begin{pmatrix} \cos \epsilon_y & 0 & -\sin \epsilon_y \\ 0 & 1 & 0 \\ \sin \epsilon_y & 0 & \cos \epsilon_y \end{pmatrix} \vec{c} . \tag{41}$$

The second rotation is around the x-axis:

$$\vec{c}^{\,x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \epsilon_x & -\sin \epsilon_x \\ 0 & \sin \epsilon_x & \cos \epsilon_x \end{pmatrix} \vec{c}^{\,y} . \tag{42}$$

The third rotation is around the z-axis, leaving the value of z_c^x unchanged:

$$\vec{c}^{\,z} = \begin{pmatrix} \cos \epsilon_z & -\sin \epsilon_z & 0 \\ \sin \epsilon_z & \cos \epsilon_z & 0 \\ 0 & 0 & 1 \end{pmatrix} \vec{c}^{\,x} . \tag{43}$$

Finally, there is the shifting of $+h_f$ along the new z-axis

$$\overrightarrow{x} = \begin{pmatrix} 0 \\ 0 \\ h_f \end{pmatrix} + \overrightarrow{c}^{z} . \tag{44}$$

Together, this gives for the new coordinates in the x-space

$$\begin{aligned} x = {} & x_c \left(\cos \epsilon_y \, \cos \epsilon_z - \sin \epsilon_x \, \sin \epsilon_y \, \sin \epsilon_z\right) \\ & - y_c \, \cos \epsilon_x \, \sin \epsilon_z \\ & + z_c \left(\sin \epsilon_y \, \cos \epsilon_z + \sin \epsilon_x \, \cos \epsilon_y \, \sin \epsilon_z\right) , \end{aligned} \tag{45}$$

$$\begin{aligned} y = {} & x_c \left(\cos \epsilon_y \, \sin \epsilon_z + \sin \epsilon_x \, \sin \epsilon_y \, \cos \epsilon_z\right) \\ & + y_c \, \cos \epsilon_x \, \cos \epsilon_z \\ & + z_c \left(\sin \epsilon_y \, \sin \epsilon_z - \sin \epsilon_x \, \cos \epsilon_y \, \cos \epsilon_z\right) , \end{aligned} \tag{46}$$

$$\begin{aligned} z = {} & -x_c \, \cos \epsilon_x \, \sin \epsilon_y + y_c \, \sin \epsilon_x + z_c \, \cos \epsilon_x \cos \epsilon_y \\ & + h_f . \end{aligned} \tag{47}$$

3 Determination of the location of a ball

In the p-frame, the diameter of the ball is measured using two end points of a diameter in any orientation, $D1$ and $D2$. The basic procedure is as the following, but additionally, it needs a correction for the real ball diameter and a correction for the real visibility of the ball.

$$\begin{aligned} D1_p &= \left(p_{D1x}, p_{D1y}\right) , \\ D2_p &= \left(p_{D2x}, p_{D2y}\right) . \end{aligned} \tag{48}$$

If the ball is rather a long line because of a high shutter speed (which is the case at the "one picture measurement"), the shortest diameter at a representative place, i.e., at the beginning or at the end of the line, has to be choosen. These two points will be transformed to the t-frame and we get t_{D1} and t_{D2}. Now, their middle point t_S and the length d_b^t of $\overline{t_{D1} t_{D2}}$ have to be calculated:

$$\begin{aligned} t_{Sx} &= \frac{t_{D1x} + t_{D2x}}{2} , \\ t_{Sy} &= \frac{t_{D1y} + t_{D2y}}{2} , \end{aligned} \tag{49}$$

$$d_b^t = \left(\left(t_{D2x} - t_{D1x}\right)^2 + \left(t_{D2y} - t_{D1y}\right)^2 \right)^{\frac{1}{2}} . \tag{50}$$

Since the real diameter d_b of the ball is known, the transformation to the c-space follows from equations 10 and 11. The distance from the focus the centre of the ball, h_S, is given with

$$d_b = d_b^t \cdot m \cdot h_S , \tag{51}$$

and the coordinates of t_S in the c-space are

$$
\begin{aligned}
Sx_c &= t_{Sx}\,\frac{d_b}{d_b^t}\,,\\
Sy_c &= t_{Sy}\,\frac{d_b}{d_b^t}\,,\\
Sz_c &= -h_S = -\frac{d_b}{d_b^t \cdot m}\,.
\end{aligned}
\tag{52}
$$

With the equations 45 to 47 we have the mapping from the coordinate system of the camera to the reference system on the ground and therefore, the location of the ball is determined.

3.1 Correction of the real ball diameter

Actually, the above mentioned calculation is only valid for a ball diameter, which is measured vertical to the line connecting the centre of the ball with the origin of the p- and t-frame. In all the other situations, the ball diameter appears too big and finally it results a too close distance to the camera. The cause of this error can be understand with figure 12: What should be measured is the real diameter of the ball, $\overline{D1r\;D2r}$. Actually, a projection $\overline{D1\;D2}$ to a ground plane G, which is parallel to the t-frame and which shall be defined to go through the centre of the ball, is measured. In the situation, where the diameter points $D1r$ and $D2r$ have the same distance from the camera, i.e., the diameter is parallel to G (and is on this ground plane, actually), $D1$ and $D1r$ are congruent.

In the figure (see as well figure 13), beside the mentioned ground plane G, two more planes are used: The D-plane, connecting the origin with the two diameter points $D1$ and $D2$, and the V-plane, which goes through the centre of the ball, C, and which is vertical to co, the line from the origin to C. dg is the intersection line between the D and G planes, dv is the intersection line between D and V, and dl is the intersection line between the G and V planes (in figure 12, dl is congruent with the point C).

For $\overline{D1r\;D2r}$, it is:

$$\overline{D1r\;D2r} = \overline{D1\;D2} \cdot \cos(\delta)\,, \tag{53}$$

where δ is the angle between dv and dg. δ can be obtained with the help of co, the line from the centre of the ball to the origin. The angle between dv and co is always $\frac{\pi}{2}$, therefore, only γ, the angle between co and dg has to be calculated. For this, first the intersection line dg has to be determined. dg is given with $D1$ and $D2$ on the G plane, for any (x, y) on dg it is

$$y = a \cdot x + c\,, \tag{54}$$

with

$$a = \frac{D2_y - D1_y}{D2_x - D1_x}\,, \tag{55}$$

$$c = \frac{D1_y \cdot D2_x - D1_x \cdot D2_y}{D2_x - D1_x}\,. \tag{56}$$

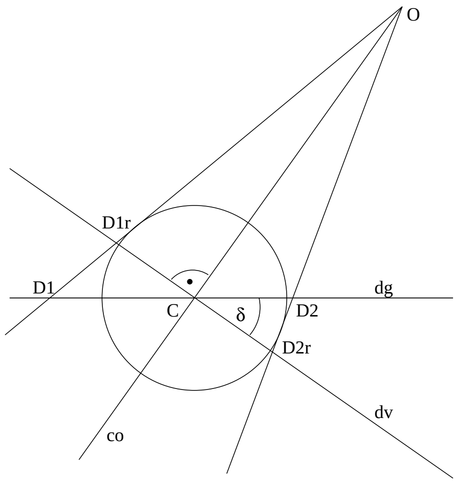

Figure 12: Representation in the D-plane, which goes through the origin and the two measured diameter pints $D1$ and $D2$. The length $\overline{D1\,D2}$ does not represent the real diameter $\overline{D1r\,D2r}$. The necessary relation is given with the angle δ, which is defined with the lines dg and dv. This lines are the intersection lines of two planes with the D-plane: G is the ground plane, which is vertical to the axis of the lens and goes through the centre of the ball, with dg as the intersection line, and V is the plane vertical to co, the line from the origin to the centre of the ball, the intersection line between D and V is dv.

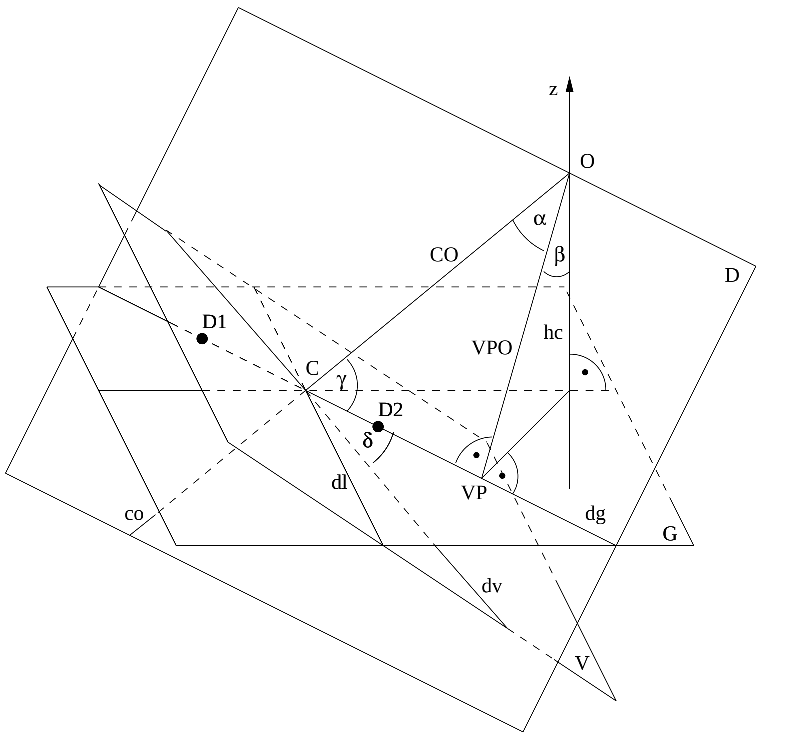

Figure 13: Calculation of the angle δ in the c-space: For this purpose, the point VP is needed and the angles γ, α and β.

On dg, we need the point $VP = (VP_x, VP_y)$, where $\overrightarrow{VPO}$ is vertical to dg. It is

$$\overrightarrow{dg} = \begin{pmatrix} D2_x - D1_x \\ D2_y - D1_y \end{pmatrix}, \tag{57}$$

$$\overrightarrow{VPO} = \begin{pmatrix} O_x - VP_x \\ O_y - VP_y \end{pmatrix}, \tag{58}$$

with $O_x = O_y = 0$ as the origin. The scalar product must be zero:

$$\langle \overrightarrow{dg}, \overrightarrow{VPO} \rangle = 0\,, \tag{59}$$

$$(D2_x - D1_x)(-VP_x) + (D2_y - D1_y)(-VP_y) = 0\,. \tag{60}$$

Together with equation 54, this gives for VP_x and VP_y of the point VP

$$VP_x = \frac{c(D2_y - D1_y)}{D1_x - D2_x + a(D1_y - D2_y)} \tag{61}$$

$$VP_y = a \cdot VP_x + c\,. \tag{62}$$

For the angle γ it is

$$\sin\gamma = \frac{\overline{VPO}}{\overline{CO}}\,, \tag{63}$$

with (for the angles, see as well figure 13)

$$\overline{CO} = \frac{hc}{\cos\alpha}\,, \tag{64}$$

$$\overline{VPO} = \frac{hc}{\cos\beta}\,. \tag{65}$$

This gives for γ and $\delta = \frac{\pi}{2} - \gamma$

$$\sin\gamma = \frac{\cos\alpha}{\cos\beta} = \cos\delta\,. \tag{66}$$

I.e., we need to know the angles between the z-axis and CO and VPO, α and β, respectively.

They can be obtained with the mapping from the t-frame to the c-space. In figure 14 it is for a point $P = (P_x, P_y, P_z)$: $P_x = t_{Px} \cdot m \cdot hp$ and $P_y = t_{Py} \cdot m \cdot hp$ (m follows from the lens geometry, equation 7).

For $dp = \overline{PO}$ it is

$$dp^2 = (t_{Px} \cdot m \cdot hp)^2 + (t_{Py} \cdot m \cdot hp)^2 + hp^2\,, \tag{67}$$

For ϕ it follows

$$\cos\phi = \frac{hp}{dp} = \frac{1}{\sqrt{m^2({t_{Px}}^2 + {t_{Py}}^2) + 1}}\,. \tag{68}$$

Finally, the angle δ is determined: With VP_x and VP_y from equation 61 and 62 and with equation 7, t_{Vx} and t_{Vy} can be calculated and the angle β is obtained. C is given with the diameter points $D1$ and $D2$, which gives the necessary angle α.

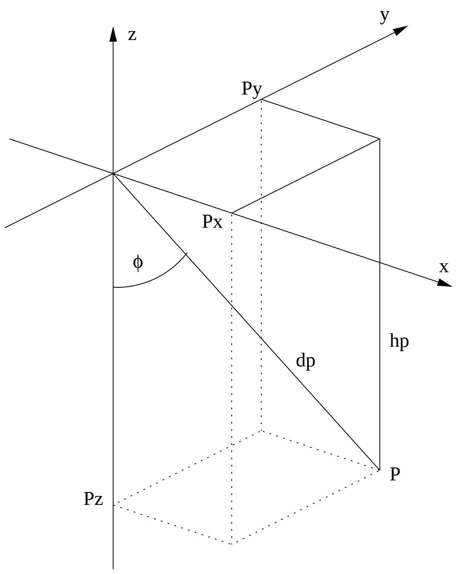

Figure 14: Determination of the angle ϕ between the z-axis and the line from the origin to a point in the c-space.

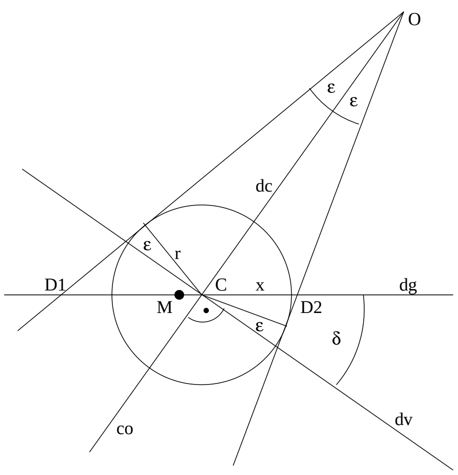

Figure 15: Effectiv location of the centre of the ball. In case of the diameter is not measured vertical to the line connecting the origin and the ball centre (i.e., where the D- and the V-planes are not congruent), M, the middle of the line from $D1$ to $D2$ is not equal to the real centre of the ball.

3.2 Effective location of the centre of the ball

In the situation, where the diameter of the ball is not measured in the direction, which is vertical to the line *co*, which goes to the origin, the centre of the ball $C = (C_x, C_y)$ is not simply in the middle of the visible diameter, $M = (M_x, M_y)$ (figure 15):

$$M_i = D1_i + \frac{D2_i - D1_i}{2} \neq C_i \tag{69}$$

$(i = x, y)$. With $x = \overline{C\,D2}$ and r as the ball diameter, it is

$$\cos(\delta - \varepsilon) = \frac{r}{x}, \tag{70}$$

$$\cos(\delta + \varepsilon) = \frac{r}{\overline{D1\,D2} - x}. \tag{71}$$

For the ratio $x/\overline{D1\,D2}$ it follows

$$\frac{x}{\overline{D1\,D2}} = \frac{\cos(\delta + \varepsilon)}{\cos(\delta - \varepsilon) - \cos(\delta + \varepsilon)}. \tag{72}$$

The angle δ is the same as in the previous section, and ε can be obtained with ($r = d_b/2$, equation 51, the derivation of dc is like in equation 67)

$$\begin{aligned} \sin\varepsilon &= \frac{r}{dc} \\ &= \frac{t_{dball} \cdot m \cdot hc/2}{hc\sqrt{(t_{Cx}\,m)^2 + (t_{Cy}\,m)^2 + 1}} \\ &= \frac{t_{dball} \cdot m}{2\left((t_{Cx}\,m)^2 + (t_{Cy}\,m)^2 + 1\right)}. \end{aligned} \tag{73}$$

Here, for t_{dball}, the values from the real ball diameter from the previous section, in *newpixel* units, have to be used, and since the real $C = (C_x, C_y)$ is not known yet, M can be used. This error can be neglected. Altogether, we get the ratio $x/\overline{D1\,D2}$, which leads to the effective ball centre between $D1$ and $D2$.

3.3 Effective visibility of the ball

From a ball, we do not see the real ball diameter $2r$, but the shorter length $2l$ (figure 16). Instead of

$$2r = t_{2r} \cdot m \cdot hr \tag{74}$$

we measure

$$2l = t_{2l} \cdot m \cdot hl\,. \tag{75}$$

The angle ε is the same as in the previous section:

$$\sin\varepsilon = \frac{t_{dball} \cdot m}{2\left((t_{Cx}\,m)^2 + (t_{Cy}\,m)^2 + 1\right)}. \tag{76}$$

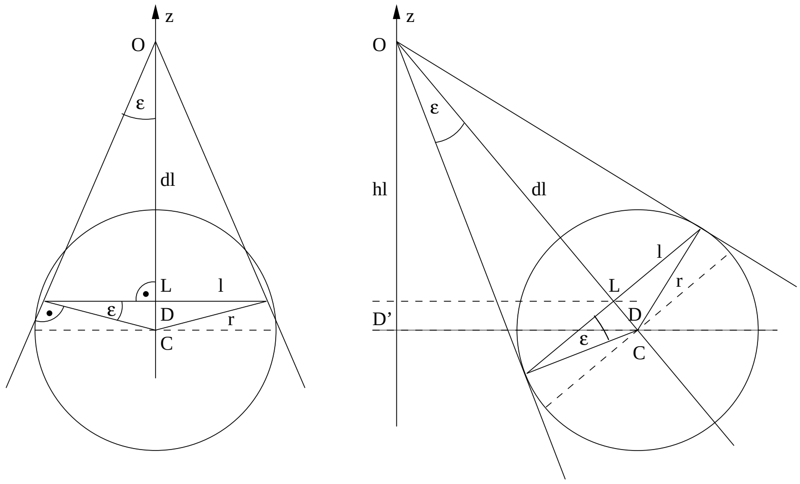

Figure 16: Real visibility of the ball diameter: Not the real diameter $2r$ is seen but a shorter length $2l$. The left picture shows a front view of the situation, the right one is a side view.

Related to ε, there follow different equations:

$$\tan\varepsilon = \frac{D}{l}, \tag{77}$$

$$\sin\varepsilon = \frac{r}{dl + D}. \tag{78}$$

Together with equation 75, this gives

$$\frac{t_{2l} \cdot m \cdot hl}{2} = \frac{r - dl\,\sin\varepsilon}{\sin\varepsilon\,\tan\varepsilon}. \tag{79}$$

The relation from dl to hl has already been used above (equation 67):

$$hl = \frac{dl}{\sqrt{(t_{Lx}\,m)^2 + (t_{Ly}\,m)^2 + 1}} \tag{80}$$

from now on, the root expression shall be abreviated with $\sqrt{\cdots}$. Finally, we get for dl

$$dl = \frac{2\,r\sqrt{\cdots}}{t_{2l} \cdot m\sin\varepsilon\,\tan\varepsilon + 2\,\sin\varepsilon\sqrt{\cdots}}. \tag{81}$$

Together with equation 78, D is obtained and therefore, with $dr = dl + D$ the real (direct, not vertical) distance of the ball is determined. To get $hr = hl + D'$, we obtain for D'

$$D' = D\,\frac{hl}{dl} = \frac{D}{\sqrt{\cdots}} \tag{82}$$

and there is, finally,

$$hr = \frac{dl + D}{\sqrt{\cdots}} \tag{83}$$

the real vertical distance from the ball to the lens. The centre of the ball has to be shifted as well, it will be slightly further away from the origin (with $i = x, y$):

$$T_{Ci} = T_{Li}\,\frac{dl + D}{dl}. \tag{84}$$

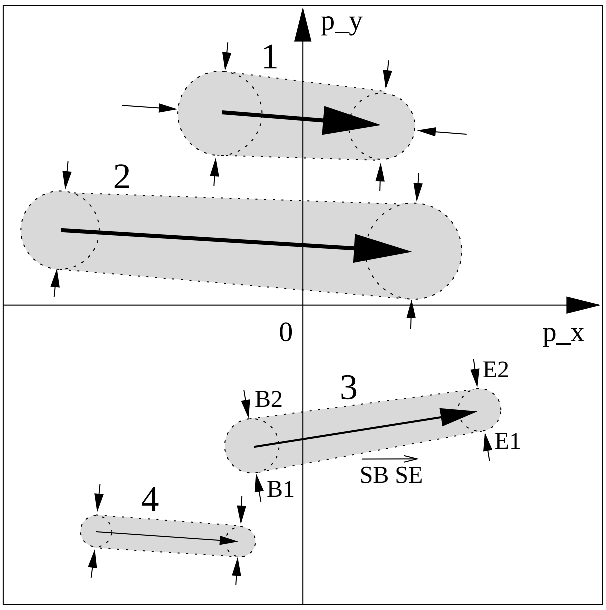

Figure 17: Representation of the one picture measurement. The shutter speed is set to a low value (such as 1/30 s) and the balls appear as a streak. The balls very close to the camera appear big (balls 1 and 2), the longer the streak, the higher the velocity (2 and 3). At each end of the faible streak the diameter, marked with two arrows, has to be measured. To get more accuracy related to the length of the streak, the end points should be measured too, as it is marked at the ball 1.

4 One picture measurement

At the one picture measurement, the velocity of a ball is measured within one frame of the video shot. Using a low shutter speed t_s, i.e., values of 1/200 to 1/30 s, the shape of one ball is not distinct but stretched to a long streak, which is related to the velocity of the ball (figure 17). The longer the streak, the higher the velocity of the ball, the larger the streak, the closer is the ball to the camera. By determining the diameters at the beginning and at the end point of the ball ($B1$, $B2$ and $E1$, $E2$ at the ball 3 in the figure), the locations S_B and S_E can be calculated, the three-dimensional velocity $\overrightarrow{v_s}$ of the ball is given with

$$\overrightarrow{v_s} = \frac{\overrightarrow{S_B S_E}}{t_s}\,. \tag{85}$$

It is easy to see: to get the velocity, the whole length of the streak has to be recognized. It follows that in the case of a dense flow, only the velocities of the balls in the upper part can be measured.

Since the ball is stretched to a long line, it looses a lot of contrast; it might be very difficult to determine the end points, especially in dense flows, where all the balls have the same color. Using different colors might be very helpful.

4.1 Characterization of the setup

In order to get a good resolution of the ball distance, the ball should be very close to the camera (see figure 1). Assuming a maximum flow height over ground h_{max} and a maximum detectable velocity v_{max}, it follows for the length of one streak l_s

$$l_s = v_{max} \cdot t_s + d_b\,. \tag{86}$$

Since the ball should be recognized in one or the following frame, $v_{max} \cdot t_i$ (t_i frame interval of the camera) has to be added to l_s. The minimum length l_{min}, which has to be recognized at the flow height of h_{max} is therefore

$$l_{min} = v_{max}\,(t_s + t_i) + d_b\,. \tag{87}$$

The length of representation $l(d_{min}) = p_{max} \cdot m \cdot d_{min}$, dependant on the distance d_{min} to the focus of the camera and the maximum number of pixels p_{max} in the video frame in the flow direction, must be equal to l_{min}. It follows for the minimum required distance from the maximum flow height to the focus of the camera

$$d_{min} = \frac{v_{max}\,(t_s + t_i) + d_b}{p_{max} \cdot m}\,. \tag{88}$$

The camera must be fixed at least at the distance $h_{max} + d_{min}$ over the ground. Here it is assumed the axis of the camera is perpendicular to the ground.

5 Two picture measurement

Here the velocity of a ball is measured with the time difference t_i from one video shot to the next one (figure 18). The shutter speed must be set very high in order to get a distinct shape of the ball ($t_s \approx 1/10000 \ldots 1/2000\,\mathrm{s}$). Beside of this, the procedure is analogous to the one picture measurement: The location of the ball is measured in the first frame, with the diameter-points $B1$ and $B2$, and in the next frame, $E1$ and $E2$.

$$\vec{v_i} = \frac{\overrightarrow{S_B S_E}}{t_i}\,. \tag{89}$$

The balls, or at least, some balls, have to be distinguished clearly from one frame to the other one. This goes better when the balls have different colors or have specific marks. Related to the "view depth", the same remarks can be added as above: the denser the flow the less deep the video camera can see.

In addition to the measurement of the velocities, single video frames made with a high shutter speed can be used for the estimation of the ball densities at different heights. Since the balls must not be tracked from one frame to the other one, it is easier to see into the flow and to determine the flow heights of the balls.

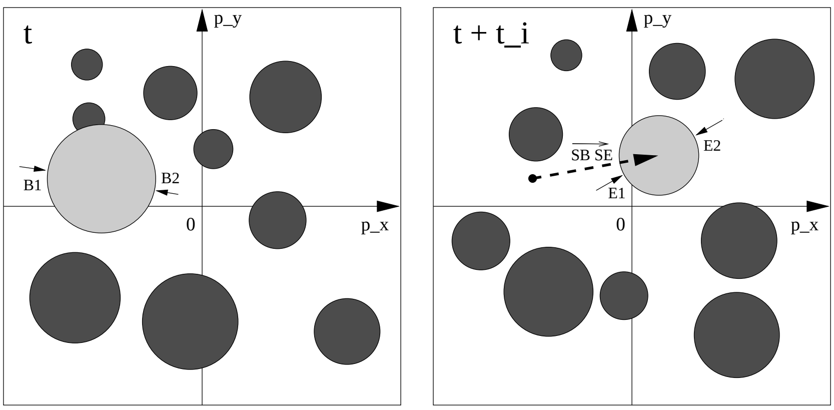

Figure 18: Velocity measurement with two video frames. The time interval from one frame to the other one is t_i, the shutter speed is set to a high value (1/10000 - 1/2000 s) to get a clear shape of the balls.

5.1 Characterization of the setup

The same equations as above are valid, only that the shutter speed t_s has to be replaced with t_i, since the balls have to be tracked during a longer time. It is:

$$l_{min} = 2 \cdot v_{max} \cdot t_i + d_b \; , \tag{90}$$

$$d_{min} = \frac{2 \cdot v_{max} \cdot t_i + d_b}{p_{max} \cdot m} \; . \tag{91}$$

It follows that the camera must be fixed a bit higher above the flow as at the one picture measurement.

6 Accuracy of the measurement

The accuracy is mostly dependent on the pixel resolution of the video frame p_{max} and on the lens characterization. Both informations are included in the constant m. Let's assume an error of Δt in pixel or newpixel units (for this calculation the difference can be neglected) at the measurement of any point in the p-plane. The error of the middle point of a ball, given from the end points of the diameter, is $\Delta t/\sqrt{2}$. Related to the diameter, the error is $(2\,\Delta t)/\sqrt{2}$. From equation 51 follows the error Δh_S in the calculation of the distance h_S from the focus of the camera:

$$\Delta h_S = \left| \frac{d_b}{(d_b^t \pm (2\,\Delta t/\sqrt{2}))\, m} - h_S \right| \; . \tag{92}$$

In dependance on the distance of the camera h_S it is

$$\Delta h_S = \left| \frac{d_b \cdot h_S}{d_b \pm (2\,\Delta t/\sqrt{2}) \cdot m \cdot h_S} - h_S \right| \; , \tag{93}$$

the result can be seen in figure 19. The error ΔS_c in the x- or y-coordinate is (equation 52)

$$\Delta S_c = \left| (t_S \pm \Delta t/\sqrt{2})\, \frac{d_b}{d_b^t \pm (2\,\Delta t/\sqrt{2})} - S_c \right| \; , \tag{94}$$

which leads to

$$\Delta S_c = \left| \frac{S_c \cdot d_b \pm (\Delta t/\sqrt{2}) \cdot m \cdot h_S \cdot d_b}{d_b \pm (2\,\Delta t/\sqrt{2}) \cdot m \cdot h_S} - S_c \right| \; . \tag{95}$$

The error ΔS_c grows from a minimum at 0 to higher values with increasing S_c (figure 20).

The error Δv_h of the horizontal velocity component is related to the beginning and end point, B_c and E_c, of the vector:

$$\Delta v_h = \frac{1}{\sqrt{2}} \left| \frac{\Delta B_c + \Delta E_c}{t_d} \right| \; , \tag{96}$$

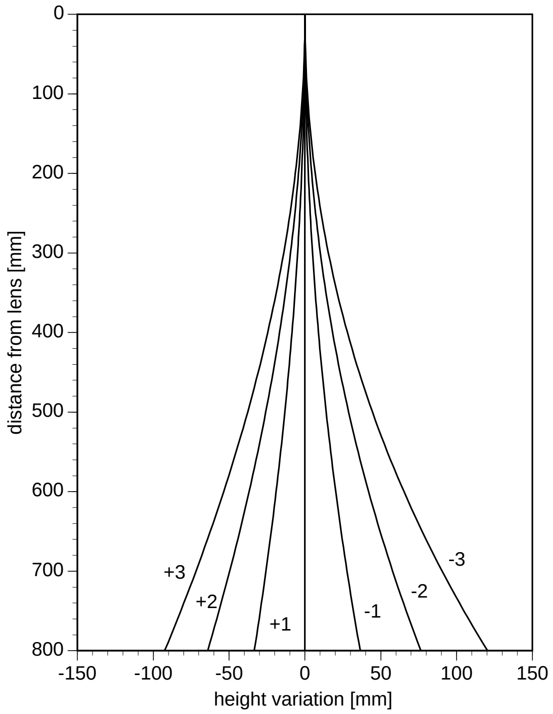

Figure 19: Error Δh_S in the calculation of the distance of the ball from the focus. The curves for the errors of $\Delta t = \pm 1, 2, 3$ in the measurement of each pixel coordinate are shown. The parameters are set to common values: $d_b = 38\,\text{mm}$ and $m = 0.00225$ (wide angle lens with additional adapter).

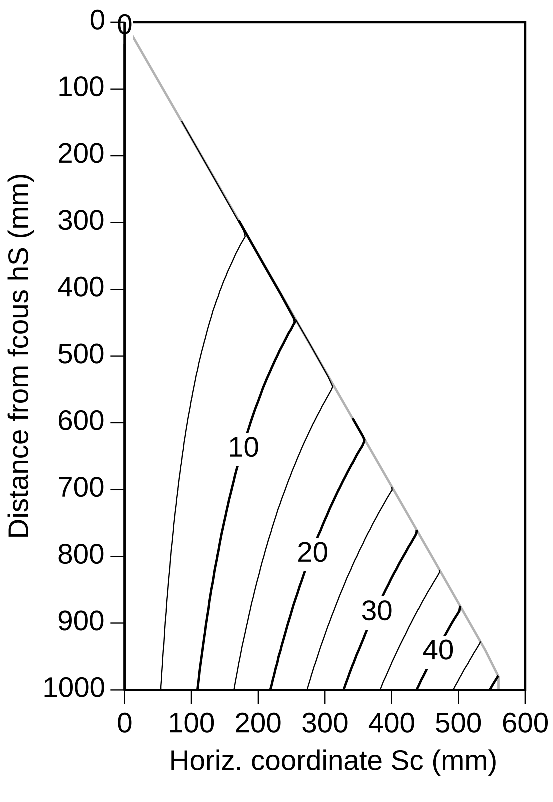

Figure 20: Error ΔS_c (in mm) in the horizontal component of the coordinate, in dependance on the horizontal coordinate and the vertical distance from the focus. The part to the upper right of the diagonal can not be seen from the camera. It is assumed, that there occurs an error of $\Delta t = \pm 1$ in the measurement of each pixel coordinate, while the other parameters are set to common values: $d_b = 38\,\text{mm}$ and $m = 0.00225$.

with t_d equals to t_s or t_i , related to the measurement method. It can be seen: The higher the time interval t_d the lower the error. For the error Δv_z of the vertical velocity component it is similar

$$\Delta v_z = \frac{1}{\sqrt{2}} \left| \frac{\Delta h_B + \Delta h_E}{t_d} \right| . \tag{97}$$

The estimation of the errors in measuring the coordinates shows very clearly the strong increasing of the inaccuracy with increasing distance from the camera and with increasing distance from the axis of the lens. As a conclusion for the measurement it follows that the balls should be as close to the camera as possible, and balls only situated near the axis of the camera should be used for the evaluation of the position and the velocity.

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